Traveling distance of random walk on UIPT
Course project for 18.677 *Stochastic Processes*

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Abstract

In this expository note, we sketch the proof [GM17, GH18] of the fact that a simple random walk on the uniform infinite planar triangulation (UIPT) typically travels with graph distance $n^{1/4+o(1)}$ in n steps. The proof builds on a detailed understanding of the mated-CRT map encoded by a SLE-decorated LQG surface, together with a strong coupling between the mated-CRT map and the UIPT [GHS20] inspired from the mating of trees theorem [DMS14].

1 Introduction

The uniform infinite planar triangulation (UIPT) $(\mathbb{M}, \mathrm{v})$, introduced in [AS03], is a random rooted infinite planar map which serves as the local limit of rooted uniform triangulations $(\mathbb{M}_N, \mathrm{v}_N)$ with N vertices as $N \to \infty$. We consider a simple random walk $\{X_j\}_{j=0}^{\infty}$ on $\mathbb{M}$ starting from v, and we are interested in the asymptotic behavior of the traveling distance $\max_{1\le j\le n} \mathrm{dist}^{\mathbb{M}}(\mathrm{v}, X_j)$ as $n \to \infty$ (here $\mathrm{dist}^{\mathbb{M}}$ denotes the graph metric on $\mathbb{M}$). It was first conjectured by [BC13] and then proved in [GM17, GH18] that the traveling distance in n steps is typically $n^{1/4+o_n(1)}$.

Theorem 1.1 ([GM17, GH18])**.** *Let $\{X_j\}_{j=1}^{\infty}$ be a SRW on $\mathbb{M}$ starting from* v*, then it holds as $n \to \infty$ that*

$$\mathbb{P}\left[\max_{1\le j\le n} \mathrm{dist}^{\mathbb{M}}(\mathrm{v}, X_j) = n^{1/4+o(1)}\right] = 1 - o(1)\,,$$

where $\mathbb{P}$ is taken with respect to the randomness of $\mathbb{M}$ and the random walk $\{X_j\}_{j=1}^{\infty}$.

Throughout this note, we write $\gamma^* = \sqrt{8/3}$, which is the LQG parameter that corresponds to the continum limit of UIPT. We note that the exponent $1/4$ in Theorem 1.1 is just $1/d_{\gamma^*}$ where d_γ is the Hausdorff dimension of the γ-LQG metric (which was shown to exist by [DG18]). We point out that an analogous result holds for some other random planar maps, with modified choices of γ; see [GM17, GH18] for these extensions.

We briefly sketch the main ingredients used in the proof. The key tool is a strong coupling between the UIPT and the so-called mated-CRT map. The mated-CRT map, a $\mathbb{C}$-embedded planar map, is encoded by a γ^*-quantum cone, along with an independent whole-plane SLE curve. This coupling allows us to shift our focus from the UIPT to the mated-CRT map. The mated-CRT map proves to be more tractable for two main reasons: (i) we can compare the graph distance with the Euclidean distance on mated-CRT maps, thanks to the comparison results in [DG18]; and (ii) the Euclidean geometry of the mated-CRT maps can be well-understood through the application of tools and results in the theory of LQG surfaces and SLE curves.

The note is organized as follows: in Section 2 we provide necessary preliminaries for the proof, and in Section 3 and Section 4, we prove the lower and upper bounds in Theorem 1.1, respectively. We also emphasize that, due to the expository purpose of this note, our focus will be on explaining the proof at the conceptual level. Most technical details will be omitted, and we may simplify certain arguments for ease of comprehension at the sacrifice of perfect precision—these instances will always be explicitly pointed out.

2 Preliminaries

In this section, we provide the essentials of the mated-CRT maps, and collect the main technical inputs for the proof of Theorem 1.1.

2.1 The mated-CRT maps

Let $(L_t, R_t)_{t\in\mathbb{R}}$ be a two-sided correlated Brownian motion on $\mathbb{R}^2$ with $L_0 = R_0 = 0$ and

$$\mathrm{Var}(L_t) = \mathrm{Var}(R_t) = |t|, \mathrm{Cov}(L_t, R_t) = |t|/2\,, \forall t \in \mathbb{R}\,. \tag{1}$$

For any $\varepsilon > 0$, we define a graph $\mathcal{G}^\varepsilon$ with vertex set $\varepsilon\mathbb{Z}$ as follows: for any $x_1 < x_2 \in \varepsilon\mathbb{Z}$, we let $(x_1, x_2) \in \mathcal{E}(\mathcal{G}^\varepsilon)$ if and only if

$$\inf_{t\in[x_1-\varepsilon,x_1]} L_t \vee \inf_{t\in[x_2-\varepsilon,x_2]} L_t \le \inf_{t\in[x_1,x_2-\varepsilon]} L_t\,,$$

or this is true when the L's are replaced by R's. We note it follows from the scaling invariance of Brownian motion, $\mathcal{G}^\varepsilon$ has the same distribution with $\mathcal{G}^1$ for any $\varepsilon > 0$, and we will denote $\mathcal{G} = \mathcal{G}^1$ for simplicity. Indeed, it is true that $\mathcal{G}^\varepsilon$ is almost surely planar, and it admits a canonical embedding into $\mathbb{C}$, as discussed below.

The construction of $\mathcal{G}^\varepsilon$ is a discretization of the so-called mating-of-trees map introduced in [DMS14]. Therefore, there is a LQG/SLE description for the map $\mathcal{G}^\varepsilon$ which we now discuss. Let $(\mathbb{C}, 0, \infty, h)$ be a γ^*-quantum cone with the circle-average embedding as defined in [DMS14, Section 4.3]. Basically, one can think this as a random measure μ_h on $\mathbb{C}$ which has a certain scaling invariant property. In addition, we consider a whole-plane space-filling SLE_6 curve η parameterized such that $\eta(0) = 0$ and $\mu_h([x_1, x_2]) = x_2 - x_1$ for any $x_1 < x_2 \in \mathbb{R}$. We define a graph $\widetilde{\mathcal{G}}^\varepsilon$ with vertex set $\varepsilon\mathbb{Z} = \varepsilon\mathbb{Z}$ such that for any $x_1 < x_2 \in \varepsilon\mathbb{Z}$, $(\eta(x_1), \eta(x_2)) \in \mathcal{E}(\mathcal{G}^\varepsilon)$ if and only if the boundaries of the two cells $\eta([x_1 - \varepsilon, x_1], \eta([x_2 - \varepsilon, x_2])$ intersect non-trivially. Quite remarkably, it turns out that $\mathcal{G}^\varepsilon$ and $\widetilde{\mathcal{G}}^\varepsilon$ has the same distribution (the correlated Brownina motion (L_t, R_t) can be realized by the length processes of the left and right boundaries of $\eta([-\infty, x]), x \in \mathbb{R}$, and $\widetilde{\mathcal{G}}^\varepsilon$ is obtained from the corresponding mating-of-trees map; see [DMS14, Section 8] for details). Therefore, we will identify $\widetilde{\mathcal{G}}^\varepsilon$ with $\mathcal{G}^\varepsilon$, and henceforth as we mention $\mathcal{G}^\varepsilon$, we implicitly mean a planar map on $\mathbb{C}$ with vertex set $\mathcal{V}(\mathcal{G}^\varepsilon) = \varepsilon\mathbb{Z}$ encoded by the pair (h, η) in the above manner.

2.2 Strong coupling

We now discuss the main tool that enables us to shift our attention from the UIPT to the (more tractable) mated-CRT maps. The main input for constructing such a coupling is the observation in [BHS19] that the UIPT can be encoded by a two-sided random walk on $\mathbb{Z}^2$ in a mating-of-trees manner (as the way $\mathcal{G}$ is encoded by the Brownian motion). It can be easily check that such a random walk has the same correlation structure with the Brownian motion defined as in (1), and thus the coupling follows from the fact that random walks can be strongly coupled with Brownian motions via the KMT theorem [KMT76].

To state the coupling result, we need to introduce several notations. For each $n \in \mathbb{N}$, we denote $\mathcal{G}_n$ for the induced subgraph of $\mathcal{G}$ on $[-n, n]_{\mathbb{Z}}$. We can also define a discrete version of $\mathcal{G}_n$ in the UIPT via the random walk path-encoding given in [BHS19]. Roughly speaking, [BHS19] constructs an explicit "bijective" mapping from a certain type of two-sided random walk paths to the UIPT (here we use quotes since the mapping therein is bijective in an almost surely sense), and thus we can identify the vertex set of $\mathbb{M}$ with $\mathbb{Z}$ via this mapping. Furthermore, the mapping restricts on the part of path on $[-n, n]_{\mathbb{Z}}$ yields a planar triangulation $\mathbb{M}_n$ with boundary $\partial\mathbb{M}_n$. Although it is not perfectly true that $\mathbb{M}_n$ equals to the subgraph of $\mathbb{M}$ induced on $[-n, n]_{\mathbb{Z}}$, we can innocuously think this is indeed the case as the only difference happens on the boundary $\partial\mathbb{M}_n$, which is negligible compared to the whole $\mathbb{M}_n$. Therefore, we can think of $\mathbb{M}_n$ as the analogue of $\mathcal{G}_n$ in the UIPT.

More precisely, we have a sequence of planar maps $\mathbb{M}_n$ with $\mathrm{v} \in V(\mathbb{M}_n)$ and such that $\mathbb{M}_n \setminus \partial\mathbb{M}_n$ embeds as a subgraph of $\mathbb{M}$ (with $\mathrm{v} \in V(\mathbb{M}_n)$ maps to $\mathrm{v} \in V(\mathbb{M})$). Moreover, we have a sequence of almost bijective mappings $\phi_n : V(\mathbb{M}_n) \to [-n.n]_{\mathbb{Z}}, \psi_n : [-n, n]_{\mathbb{Z}} \to V(\mathbb{M}_n)$ with $\phi_n(\mathrm{v}) = 0, \psi_n(0) = \mathrm{v}$, and $\psi_n \circ \phi_n \approx \mathrm{Id}_{V(\mathbb{M}_n)}, \phi_n \circ \psi_n \approx \mathrm{Id}_{[-n,n]_{\mathbb{Z}}}$ in a sense that will become clear in item (iii) of the following theorem.

Theorem 2.1 ([GHS20], Strong coupling between mated-CRT and UIPT)**.** *Let $\mathcal{G}_n, \mathbb{M}_n, \phi_n, \psi_n$ be defined as above, and note that $\phi_n : V(\mathbb{M}_n) \to \mathcal{V}(\mathcal{G}_n), \psi_n : \mathcal{V}(\mathcal{G}_n) \to V(\mathbb{M}_n)$. There are universal constants $C, p, q > 0$ such that there exists a coupling between $\mathcal{G}$ and $(\mathbb{M}, \mathrm{v})$ with the property that for any $n \in \mathbb{N}$, with probability $1 - O(n^{-10})$, the following hold:*

(i) For any adjacent $u, v \in V(\mathbb{M}_n)$, there exists a path $\mathcal{P}^{\mathcal{G}_n}_{u,v}$ on $\mathcal{G}_n$ from $\phi_n(u)$ to $\phi_n(v)$ with length at most $C(\log n)^p$. Moreover, each edge in $\mathcal{G}_n$ is traversed by at most $C(\log n)^q$ many such paths.

(ii) For any adjacent $x, y \in \mathcal{V}(\mathcal{G}_n)$, there exists a path $\mathcal{P}^{\mathbb{M}_n}_{x,y}$ on $\mathbb{M}_n$ from $\psi_n(x)$ to $\psi_n(y)$ with length at most $C(\log n)^p$. Moreover, each edge in $\mathbb{M}_n$ is traversed by at most $C(\log n)^q$ many such paths.

(iii) It holds that for any $v \in V(\mathbb{M}_n)$, $\mathrm{dist}^{\mathbb{M}_n}(\psi_n(\phi_n(v)), v) \le C(\log n)^p$, and for any $x \in \mathcal{V}(\mathcal{G}_n)$, $\mathrm{dist}^{\mathcal{G}_n}(\phi_n(\psi_n(x)), x) \le C(\log n)^p$, where $\mathrm{dist}^{\mathbb{M}_n}, \mathrm{dist}^{\mathcal{G}_n}$ is the graph distance on $\mathbb{M}_n, \mathcal{G}_n$, respectively.

2.3 Comparing graph metric with Euclidean metric

In light of Theorem 2.1, it is convincing that a good understanding of SRW on the mated-CRT maps also sheds light on the SRW on UIPT. The virtue of working on the mated-CRT map is that the graph metric on $\mathcal{G}$ has an explicit relation with the Euclidean metric under the canonical embedding $\eta : \mathbb{Z} \to \mathbb{C}$ defined as before, which we now discuss.

For any $\varepsilon > 0$, recall the definition of $\mathcal{G}^\varepsilon$. For any domain $D \subset \mathbb{C}$, we define $\mathcal{G}^\varepsilon(D)$ as the subgraph of $\mathcal{G}^\varepsilon$ induced by the vertices $x \in \mathcal{V}(\mathcal{G}^\varepsilon) = \varepsilon\mathbb{Z}$ such that $\eta([x - \varepsilon, x]) \cap D \neq \emptyset$ (and we view $\mathcal{G}^\varepsilon(D)$ as the planar map in $\mathbb{C}$ embedded by η). In view of the scaling invariance of $\mathcal{G}^\varepsilon, \varepsilon > 0$, the relation between graph metric and Euclidean metric of $\mathcal{G}$ is captured by the following result, which is a restatement of [DG18, Proposition 4.6].

Theorem 2.2 ([DG18], Comparison between graph-metric balls and Euclidean balls)**.** *For any $\zeta > 0$, there exists $\alpha = \alpha(\zeta) > 0$, such that for any $\varepsilon > 0$, with probability $1 - O(\varepsilon^\alpha)$, it holds that*

$$\mathcal{B}^{\mathcal{G}^\varepsilon}_{\varepsilon^{-1/4+\zeta}}(0) \subset \mathcal{V}\big(\mathcal{G}^\varepsilon(B_{1/2}(0))\big) \subset \mathcal{B}^{\mathcal{G}^\varepsilon}_{\varepsilon^{-1/4-\zeta}}(0)\,, \tag{2}$$

where $\mathcal{B}_r^{\mathcal{G}^\varepsilon}(0)$ denotes the metric ball on $\mathcal{G}^\varepsilon$ centered at 0 with radius r, and $B_r(0)$ denotes the Euclidean ball centered at 0 with radius r.

3 Proof of the lower bound

This section devotes to proving the lower bound of Theorem 1.1, that is, a SRW travels at least distance $n^{1/4-o(1)}$ in n steps with high probability. The main ingredient is a Euclidean resistance estimate on the mated-CRT maps. After proving this, we translate the estimate to the graph metric resistance of the UIPT via Theorem 2.1 and Theorem 2.2. We then obtain the desired result from standard arguments of reversible Markov chains.

In Section 3.1 we give the basic of resistance on electronic networks, then state the estimate for the Euclidean resistance on mated-CRT maps and sketch the proof. In Section 3.2, we obtain estimation for the graph metric resistance on the UIPT from results in Section 3.1 via the strong coupling. In Section 3.3, we complete the proof of the lower bound by bounding the expected exit time of the graph-metric ball.

3.1 Resistance estimate: the Euclidean setting

Consider a locally-finite graph G, let X^G be the SRW on G. For $x \in V(G)$ and $V \subset V(G)$ such that $x \notin V$ and $V(G) \setminus (\{x\} \cup V)$ is finite, we define the resistance between x and V as

$$\mathcal{R}^G(x \leftrightarrow V) = \deg^G(x)^{-1}\mathbb{E}_x[\#\{\text{times } X^G \text{ returns back to } x \text{ before hitting } V] = \deg^G(x)^{-1}\,\mathrm{Gr}^G_{\tau_V}(x,x)\,,$$

where τ_V is the first hitting time of V and $\mathrm{Gr}^G_{\tau_V}$ is the Green function for the random walk killed at the stopping time τ_V.

We will need the following two variational characterization of the resistance. The first approach involves the *Dirichlet energy*.For a function $f : V(G) \to \mathbb{R}$, define its Dirichlet energy as

$$\mathrm{Energy}(f_V; G) = \sum_{(x,y)\in E(G)} (f(x) - f(y))^2\,. \tag{3}$$

Dirichlet's principle states that $\mathcal{R}^G(x \leftrightarrow V)$ equals to $\sup \mathrm{Energy}(f_V; G)^{-1}$, where the supremum is taken over all functions $f_V : V \to \mathbb{R}$ such that $f(x) = 1$ and $f(y) = 0, \forall y \in V$ (Indeed, the maximum is achieved by the discrete harmonic function on $V(G) \setminus (\{x\} \cup V)$).

The second variational characterization takes in terms of the *unit flows*. A unit flow from x to V is a real function θ on the set of directed edges in G satisfying that $\theta(y,z) = -\theta(z,y)$ for any $(y,z) \in E(G)$ and

$$\sum_{z\sim y}\theta(y,z) = 0, \forall y \in V(G) \setminus (\{x\} \cup V)\,, \quad \text{and} \quad \sum_{y\sim x}\theta(x,y) = 1\,.$$

By the Thomson's principle, we have that

$$\mathcal{R}^G(x \leftrightarrow V) = \inf\left\{\sum_{e\in E(G)\setminus E(V)}]\theta(e)^2 : \theta \text{ is a unit flow from } x \text{ to } V\right\}\,. \tag{4}$$

Recall the definition of $\mathcal{G}^\varepsilon, \mathcal{G}^\varepsilon(D)$ for $\varepsilon > 0$ and $D \subset \mathbb{C}$. Also recall that $B_r(0)$ is the ball centered at 0 with radius r, and we write $\mathcal{V}_r^\varepsilon = \mathcal{V}\big(\mathcal{G}^\varepsilon(\mathbb{C} \setminus B_r(0))\big)$ for simplicity. The following proposition estimates the resistance between 0 and $\mathcal{V}^\varepsilon_{1/2}$ on the graph $\mathcal{G}^\varepsilon$.

Proposition 3.1 ([GM17], Proposition 3.4)**.** *There exists $\alpha, C > 0$ such that for any $\varepsilon > 0$, it holds that*

$$\mathbb{P}[\mathcal{R}^{\mathcal{G}^\varepsilon}(0 \leftrightarrow \mathcal{V}^\varepsilon_{1/2}) \le C \log \varepsilon^{-1}] \le 1 - O\left(\frac{1}{(\log \varepsilon^{-1})^\alpha}\right).$$

Below we sketch the proof of Proposition 3.1. In the remaining of this section, we will always condition on the realization of (h, η) (hence also the random planar map $\mathcal{G}^\varepsilon$). We will argue that provided (h, η) satisfies certain nice conditions (which holds true with probability $1 - O(1/(\log \varepsilon^{-1})^\alpha)$), then $\mathcal{R}^{\mathcal{G}^\varepsilon}(0 \leftrightarrow \mathcal{V}^\varepsilon_{1/2})$ is bounded by $C \log \varepsilon^{-1}$. In light of Thomson's principle (4), it suffices to construct a unit flow from 0 to $\mathcal{V}^\varepsilon_{1/2}$ that has energy (defined as in (4)) bounded by $C \log \varepsilon^{-1}$ under typical instances of (h, η). We now state the construction given in [GM17].

We define a random path on $\mathcal{G}^\varepsilon$ from 0 to $\mathcal{V}^\varepsilon_{1/2}$ as follows: pick a uniform point $\mathtt{t}$ on $\partial B_{1/2}(0)$, take the line segment $\mathtt{L}$ joining 0 to $\mathtt{t}$, and let $\mathcal{P} = \{0 = x_0, x_1, \dots, x_T\}$ be the (directed) path on $\mathcal{G}^\varepsilon$ such that $\eta([x_i - \varepsilon, x_i]), i = 0, 1, \dots, T$ are exactly all the cells that intersect $\mathtt{L}$. It is clear that $x_T \in \mathcal{V}$ since x_T corresponds to the cell that contains $\mathtt{t}$. For a directed edge e in $\mathcal{G}^\varepsilon$, the flow function is defined as

$$\theta(e) = \mathbb{P}[\mathcal{P} \text{ traverses } e] - \mathbb{P}[\mathcal{P} \text{ traverses } \bar{e}] \tag{5}$$

where $\bar{e}$ is the reverse of e. Here and below $\mathbb{P}$ is only the randomness coming from the choice of $\mathtt{t}$, and we will always implicitly work under the conditioning of (h, η).

Lemma 3.2. *The function θ defined as in (5) is a unit flow from 0 to $\mathcal{V}^\varepsilon_{1/2}$.*

Proof. θ clearly satisfies that $\theta(e) = -\theta(\bar{e})$ for any e. In addition, for any $x \in \mathcal{V}(\mathcal{G}^\varepsilon)$, it holds that

$$\sum_{y \sim x} \theta(x, y) = \mathbb{E}\left[\#\{y : (x, y) \text{ is traversed by } \mathcal{P}\} - \#\{y : (y, x) \text{ is traversed by } \mathcal{P}\}\right],$$

where the expectation is taken over the random path $\mathcal{P}$. We note that for $x = 0$, the quantity in the expectation almost surely equals to $1 - 0 = 1$, and for $x \in \mathcal{V}(\mathcal{G}^\varepsilon) \setminus (\{x\} \cup \mathcal{V}^\varepsilon_{1/2})$, the quantity almost surely equals to $1 - 1 = 0$. This verifies that θ is a unit flow from x to $\mathcal{V}^\varepsilon_{1/2}$. □

By Thomson's principle, Proposition 3.1 follows once we prove that

$$\sum_{e \in \mathcal{E}(\mathcal{G}^\varepsilon) \setminus \mathcal{E}(\mathcal{V}^\varepsilon_{1/2})} \theta(e)^2 \le C \log \varepsilon^{-1} \tag{6}$$

holds for typical realizations of (h, η). In what follows we explain in detail that a weaker version of (6) holds true. Denote $A_{r_1, r_2}(0)$ for the annulus centered at 0 with inner and outer radius r_1, r_2, respectively. Then it follows that for some universal constants $\alpha, q > 0$, with probability $1 - O(\varepsilon^\alpha)$ on (h, η),

$$\sum_{e \in \mathcal{E}(\mathcal{G}^\varepsilon(A_{\varepsilon^q, 1/2}(0)))} \theta(e)^2 \le C \log \varepsilon^{-1}. \tag{7}$$

To facilitate the proof of (7), we need an additional technical input that for some universal constants $\alpha, q' > 0$, with probability $1 - O(\varepsilon^\alpha)$, the maximal diameter of the cell $\eta([x, x - \varepsilon])$ that lies in $B_{1/2}(0)$ is bounded by $\varepsilon^{q'}$ (see e.g. [GMS19, Lemma 2.4]). Conditionally on a realization of (h, η) with such an event happens, recalling that $\mathtt{L}$ is the line segment joining 0 to the uniformly random point $\mathtt{t} \in \partial B_{1/2}(0)$

and taking q to be slightly smaller than q', we see that for any edge $e = (x, y)$ in $\mathcal{G}^\varepsilon(A_{\varepsilon^q,1/2}(0))$, one has (below C denotes for some universal constant that might change from line to line)

$$\begin{aligned}
|\theta(e)| &\le \mathbb{P}[\mathtt{L} \cap \eta([x-\varepsilon, x] \cup [y-\varepsilon, y]) \neq \emptyset] && \text{(by the definition of } \theta(e)) \\
&\le C\left(\frac{\operatorname{diam}(\eta([x-\varepsilon,x]))}{\operatorname{dist}(\eta([x-\varepsilon,x]),0)} + \frac{\operatorname{diam}(\eta([y-\varepsilon,y]))}{\operatorname{dist}(\eta([y-\varepsilon,y]),0)}\right) && \text{(by elementary Euclidean geometry)} \\
&\le C\left(\frac{\operatorname{diam}(\eta([x-\varepsilon,x]))}{|\eta(x)| - \varepsilon^{q'}} + \frac{\operatorname{diam}(\eta([y-\varepsilon,y]))}{|\eta(y)| - \varepsilon^{q'}}\right) && \text{(by the cell-diameter control)} \\
&\le C\left(\frac{\operatorname{diam}(\eta([x-\varepsilon,x]))}{|\eta(x)|} + \frac{\operatorname{diam}(\eta([y-\varepsilon,y]))}{|\eta(y)|}\right). && \text{(by the choice that } x, y \in \mathcal{V}(\mathcal{G}^\varepsilon(A_{\varepsilon^q,1/2}(0))))
\end{aligned}$$

Therefore, we conclude that under such a nice event, the left hand side of (7) is upper-bounded by

$$\begin{aligned}
&C \sum_{(x,y)\in\mathcal{E}(\mathcal{G}^\varepsilon(A_{\varepsilon^q,1/2}(0)))} \left[\left(\frac{\operatorname{diam}(\eta([x-\varepsilon,x]))}{|\eta(x)|}\right)^2 + \left(\frac{\operatorname{diam}(\eta([y-\varepsilon,y]))}{|\eta(y)|}\right)^2\right] \\
&\le C \sum_{x\in\mathcal{V}(\mathcal{G}^\varepsilon(A_{\varepsilon^q,1/2}(0)))} \deg^{\mathcal{G}^\varepsilon}(x) \left(\frac{\operatorname{diam}(\eta([x-\varepsilon,x]))}{|\eta(x)|}\right)^2 .
\end{aligned}$$

Ignoring the degree term, this appears very akin to a Riemann sum, and hence it is reasonable to expect something like

$$\sum_{x\in\mathcal{V}(\mathcal{G}^\varepsilon(A_{\varepsilon^q,1/2}(0)))} \left(\frac{\operatorname{diam}(\eta([x-\varepsilon,x]))}{|\eta(x)|}\right)^2 \lesssim \int_{A_{\varepsilon^q,1/2}(0)} \frac{\mathrm{d}x}{|x|^2} \le C \log \varepsilon^{-1} .$$

Moreover, one can expect that the additional degree term should not make things much worse, as it is known that the degrees (as random variables) are typically of order $O(1)$ and they all have exponential tails. Intuitively, this gives the desired estimate (7), and indeed this can be made rigorous by applying [GMS19, Lemma 3.1]. This concludes that (7) holds with high probability.

We conclude this section by a few words about how can one improve (7) to (6). Very roughly speaking, this follows from a multi-scale analysis that takes advantage of the scaling property of the quantum cone. On a very heuristic level, one can think the energy in each scale $A_{\varepsilon^{kq},\varepsilon^{(k-1)q}}(0), k = 1, 2, \dots, \lceil q^{-1} \rceil$ is approximately the same due to the scaling property, and thus (6) holds with C taken roughly as $q^{-1}C'$ where C' is the constant appearing in (7). The actual proof of (6) requires significantly more effort, and we refer the interested readers to Section 3.3 of [GM17] for more details.

3.2 Resistance estimate: the UIPT setting

Recall that $(\mathbb{M}, \mathrm{v})$ is the rooted UIPT. For $r \in \mathbb{N}$, we let $\mathcal{B}_r^{\mathbb{M}}(\mathrm{v})$ be the graph metric ball on $\mathbb{M}$ centered at v with radius r. The main goal of this subsection is to prove the following resistance estimate for the UIPT:

Proposition 3.3. *There are universal constants $p, \alpha, C > 0$ such that for any $r \in \mathbb{N}$, it holds with probability $1 - O((\log r)^{-\alpha})$ that*

$$\mathcal{R}^{\mathbb{M}}(\mathrm{v} \leftrightarrow (\mathcal{B}_r^{\mathbb{M}})^c) \le C(\log r)^p .$$

Proof. The proof of Proposition 3.3 contains two main steps. The first is to derive a graph-metric ball resistance estimate of the mated-CRT maps, and the second is to translate this estimate to the UIPT case. The first step follows from Proposition 3.1 and the comparison result Theorem 2.2. The second step is achieved by the strong coupling between mated-CRT maps and the UIPT (Theorem 2.1). We now elaborate these two steps more precisely.

Step 1: from Euclidean metric to graph metric. Recall that as in Theorem 2.2, we denote $\mathcal{B}_r^{\mathcal{G}^\varepsilon}(0)$ for the graph metric ball on $\mathcal{G}^\varepsilon$ centered at 0 with radius r. Fix some small constant $\zeta > 0$, for any $r \in \mathbb{N}$, we let $\varepsilon = \varepsilon(r)$ be such that $\varepsilon^{-1/4+\zeta} = r$. Since $\mathcal{G}$ has the same distribution with $\mathcal{G}^\varepsilon$, we have that

$$\mathcal{R}^{\mathcal{G}}(0 \leftrightarrow (\mathcal{B}_r^{\mathcal{G}}(0))^c) \stackrel{\mathrm{d}}{=} \mathcal{R}^{\mathcal{G}^\varepsilon}(0 \leftrightarrow (\mathcal{B}^{\mathcal{G}^\varepsilon}_{\varepsilon^{-1/4+\zeta}})^c)\,.$$

However, from Theorem 2.2 we know that for some $\alpha = \alpha(\zeta) > 0$, with probability $1 - O(\varepsilon^\alpha)$ it holds that $(B^{\mathcal{G}^\varepsilon}_{\varepsilon^{-1/4+\zeta}}(0))^c \supset \mathcal{V}^\varepsilon_{1/2}$ (recall that $\mathcal{V}^\varepsilon_{1/2} = (\mathcal{V}(\mathcal{G}^\varepsilon(B_{1/2}(0))))^c$). Then it follows from Proposition 3.1 together with the inclusion-monotonicity of resistance that, with probability

$$1 - O(\varepsilon^\alpha) - O(1/(\log \varepsilon^{-1})^\alpha) = 1 - O(1/(\log r)^\alpha)\,,$$

it holds that

$$\mathcal{R}^{\mathcal{G}}(0 \leftrightarrow (\mathcal{B}_r^{\mathcal{G}}(0))^c) \le C \log r\,. \tag{8}$$

Step 2: From mated-CRT maps to the UIPT. Recall the maps $\mathbb{M}_n$ and the mappings $\phi_n : V(\mathbb{M}_n) \to \mathcal{V}(\mathcal{G}_n), \psi_n : \mathcal{V}(\mathcal{G}_n) \to V(\mathbb{M}_n)$ defined as in Section 2.2. We will need the following technical fact: there exists $K > 0$ such that with probability $1 - O(r^{-1})$ it holds that $\mathcal{B}_r^{\mathbb{M}}(\mathrm{v}) = \mathcal{B}_r^{\mathbb{M}_R}(\mathrm{v})$, where $R = r^K$. In light of this, Proposition 3.3 follows provided that $\mathcal{R}^{\mathbb{M}_R}(0 \leftrightarrow (\mathcal{B}_r^{\mathbb{M}_R}(\mathrm{v}))^c) \le C(\log r)^p$ happens with probability $1 - O(1/(\log r)^\alpha)$ for some $C, p, \alpha > 0$, which we argue as below.

Let $(\mathcal{G}, \mathbb{M})$ be jointly sampled from the coupling in Theorem 2.1. Assume that the pair $(\mathcal{G}, \mathbb{M})$ satisfies that: (a) item (i)-(iii) in Theorem 2.1 holds for $n = R$, and (b) (8) holds for r replaced by $R' = CK^pR(\log r)^p$ (where C, p are the constants appearing in Theorem 2.1). We also note this happens with probability

$$1 - O(r^{-10K}) - O(1/(\log R')^\alpha) = 1 - O(1/(\log r)^\alpha)\,.$$

Recall the Dirichlet energy characterization of resistance (3). For any function $f : V(\mathbb{M}_R) \to \mathbb{R}$ satisfying that $f(\mathrm{v}) = 1$ and $f(u) = 0, \forall u \notin \mathcal{B}_r^{\mathbb{M}_R}(\mathrm{v})$, define an associate function $g : \mathcal{V}(\mathcal{G}_R) \to \mathbb{R}$ as $g = f \circ \psi_n$. It is clear that $g(0) = 1$, and we conclude from Item (ii) in Theorem 2.1 that $g(x) = 0$ for all $x \notin \mathcal{B}^{\mathcal{G}}_{R'}(0)$. Therefore, it follows from (8) (for r replaced by R') and Dirichlet's principle that $\mathrm{Energy}(g; \mathcal{G}) \ge C^{-1}(\log r)^{-1}$. On the other hand, it is straightforward to check that Item (i) and Item (iii) in Theorem 2.1 together imply $\mathrm{Energy}(f; \mathbb{M}_R) \ge C^{-1}(\log r)^{-p-q}\,\mathrm{Energy}(g; \mathcal{G})$. Applying Dirichlet's principle once again, this yields that $\mathcal{R}^{\mathbb{M}_R}(0 \leftrightarrow (\mathcal{B}_r^{\mathbb{M}_R}(\mathrm{v}))^c) \le C^2(\log r)^{p+q+1}$, which gives the desired estimate and thus completes the proof of Proposition 3.3. $\square$

3.3 Bounding the exit time of $\mathcal{B}_r^{\mathbb{M}}(\mathrm{v})$

We conclude the lower bound in Theorem 1.1 from the next proposition together with Markov inequality.

Proposition 3.4. *For any $\zeta > 0$, let $\tau_{n,\zeta}$ be the exit time of $\mathcal{B}^{\mathbb{M}}_{n^{1/4-\zeta}}(\mathrm{v})$ of the SRW on $\mathbb{M}$. Then for any $\zeta > 0$, it holds for sufficiently large n that with probability $1 - o(1)$ over $(\mathbb{M}, \mathrm{v})$, $\mathbb{E}_{\mathrm{v}}[\tau_{n,\zeta}] = o(n)$.*

Proof. Fix $n, \delta > 0$, and write $r = n^{1/4-\delta}$ and $\tau = \tau_{n,\delta}$ for brevity. Note that for any $u \in \mathcal{B}_r^{\mathbb{M}}(\mathrm{v})$,

$$\mathrm{Gr}^{\mathbb{M}}_\tau(\mathrm{v}, u) = \frac{\deg^{\mathbb{M}}(u)}{\deg^{\mathbb{M}}(\mathrm{v})}\,\mathrm{Gr}^{\mathbb{M}}_\tau(u, \mathrm{v}) \le \deg^{\mathbb{M}}(u) \cdot \frac{\mathrm{Gr}^{\mathbb{M}}_\tau(\mathrm{v}, \mathrm{v})}{\deg^{M}(\mathrm{v})} = \deg^{\mathbb{M}}(u)\mathcal{R}^{\mathbb{M}}(0 \leftrightarrow (\mathcal{B}_r^{\mathbb{M}}(\mathrm{v}))^c)\,.$$

Summing over $u \in \mathcal{B}_r^{\mathbb{M}}(\mathrm{v})$, we get that $\mathbb{E}[\tau] \le \mathcal{R}^{\mathbb{M}}(0 \leftrightarrow (\mathcal{B}_r^{\mathbb{M}}(\mathrm{v}))^c) \cdot 2\#\mathcal{B}_r^{\mathbb{M}}(\mathrm{v})$, which is bounded by $C(\log n)^p \cdot n^{1-4\zeta+o(1)} = o(n)$ with high probability for large n, by Proposition 3.3 and a standard volume estimate for the UIPT as in [Ang03, Theorem 1.2]. This completes the proof. $\square$

4 Proof of the upper bound

In this section, we provide an outline of the proof in [GH18] that with high probability, the SRW on the UIPT travels graph distance at most $n^{1/4+o_n(1)}$ steps in n units of time, thus completing the proof of Theorem 1.1. This is done by first composing the rough isometry $\mathbb{M} \to \mathcal{G}^\epsilon$ provided by Theorem 2.1 with the embedding $x \mapsto \eta(x)$ of $\mathcal{G}^\epsilon$ to obtain an embedding of (a large subgraph of) $\mathbb{M}$ into $\mathbb{C}$. One important advantage of this particular embedding is that the graph distance diameter (with respect to $\mathbb{M}$ or $\mathcal{G}^\epsilon$) of the resulting set of vertices contained in a fixed Euclidean ball is with high probability at most $\epsilon^{-1/4+o_\epsilon(1)}$. Under this embedding, we assign a weight to each vertex that is roughly equal to the diameter of the corresponding space-filling SLE cell, so that the resulting weighted graph distance approximates the Euclidean distance. Section 4.3 gives an upper bound on the displacement of the SRW on $\mathbb{M}$ with respect to the weighted graph distance and thus the Euclidean distance using Markov-type theory in Section 4.1. Finally, the comparison between graph distance balls and Euclidean balls will complete the proof of the upper bound in Theorem 1.1. We state the result here for later reference.

Theorem 4.1. *Let $(\mathbb{M}, \mathbf{v})$ be the UIPT and let γ^* be the corresponding LQG parameter, so that $d_{\gamma^*} = 4$. For each $\zeta \in (0,1)$, there exists $\alpha = \alpha(\zeta) > 0$ so that for each $n \in \mathbb{N}$, the simple random walk $X^{\mathbb{M}}$ on $\mathbb{M}$ satisfies*

$$\mathbb{P}\left[\max_{1\le j\le n} \mathrm{dist}^{\mathbb{M}}(X_j^{\mathbb{M}}, \mathbf{v}) \le n^{1/4+\zeta}\right] \ge 1 - O_n(n^{-\alpha}).$$

4.1 Markov-type inequality

A key step in the proof of Theorem 4.1 is to apply certain Markov-type inequalities proved in [DLP13]. We say that a metric space $\mathbb{X} = (\mathbb{X}, d)$ has *maximal Markov-type* p if there exists a constant C such that the following condition holds: for every finite set S, every transition matrix P of an irreducible reversible Markov chain on S, and every function $\phi : S \to \mathbb{X}$, we have that

$$\mathbb{E}\left[\max_{0\le m\le n} d(\phi(X_0), \phi(X_m))^p\right] \le C^p n \mathbb{E}\left[d(\phi(X_0), \phi(X_1))^p\right].$$

We denote the optimal choice of C by M_p. A straightforward modification of the proof in [DLP13] implies the following result.

Proposition 4.2. *There exists a universal constant C such that every vertex-weighted planar graph has maximal Markov-type 2 with $M_2 \le C$.*

Our goal is to use the preceding proposition to obtain a diffusivity estimate for the SRW on the UIPT, i.e., when $p = 2, S = \mathbb{X} = \mathcal{V}(\mathbb{M})$ is the vertex set of the UIPT, ϕ is the identity function, and X is the SRW on $\mathbb{M}$. To do so, we must take a slight detour to introduce the notions of unimodularity and reversibility for random rooted graphs. We recall that given a weighted graph (G, ω) and vertices $v, w \in \mathcal{V}(G)$, the *weighted graph distance* is defined by

$$\mathrm{dist}_\omega^G(v, w) := \inf_P \sum_{i=1}^{|P|} \frac{1}{2}(\omega(P(i)) + \omega(P(i-1))), \tag{9}$$

where $|P|$ is the length of the path, and the infimum is over all finite paths P in G from v to w.

Definition 4.3. Let $(G, \omega, \mathbf{v})$ be a triple consisting of a connected locally finite graph, a nonnegative weighting on the vertex set of G, and a marked vertex of G, respectively. We say that $(G, \omega, \mathbf{v})$ is a *unimodular vertex-weighted graph* if it satisfies the *mass transport principle*: for each nonnegative Borel measurable function F on the space of vertex-weighted graphs with two marked points (whose topology is taken to be a natural generalization of the Benjamini-Schramm local topology [BS01]),

$$\mathbb{E}\left[\sum_{u\in\mathcal{V}(G)} F(G,\omega,\mathbf{v},u)\right] = \mathbb{E}\left[\sum_{u\in\mathcal{V}(G)} F(G,\omega,u,\mathbf{v})\right].$$

Intuitively, this is saying that at the root vertex, "the expected mass coming in equals the expected mass coming out." Unweighted unimodular random rooted graphs are defined similarly.

Definition 4.4. Let $(G, \omega, \mathbf{v})$ be as in the preceding definition. We say that $(G, \omega, \mathbf{v})$ is a *reversible vertex-weighted graph* if the following condition holds. Let $\widetilde{\mathbf{v}}$ be sampled uniformly from the set of neighbors of $\mathbf{v}$ in G. Then $(G, \omega, \mathbf{v}, \widetilde{\mathbf{v}}) \stackrel{d}{=} (G, \omega, \widetilde{\mathbf{v}}, \mathbf{v})$.

Remark 4.5. It follows from these definitions that if $(G, \omega, \mathbf{v})$ is unimodular and satisfies $\mathbb{E}[\deg(\mathbf{v})] < \infty$, then biasing the law of $(G, \omega, \mathbf{v})$ by $\deg(\mathbf{v})$ produces a random rooted graph which is reversible. Conversely, if $(G, \omega, \mathbf{v})$ is reversible, then biasing its law by $\deg^{-1}(\mathbf{v})$ produces a unimodular random rooted graph.

Recall that a *percolation* ρ on a unimodular random rooted graph $(G, \mathbf{v})$ is a random subgraph of G, where an edge is labeled 1 if it is included in the subgraph and 0 otherwise, so that the connected component $K_\rho(\mathbf{v})$ of $\mathbf{v}$ is unimodular. A percolation is said to be *finitary* if $K_\rho(\mathbf{v})$ is almost surely finite. A unimodular random rooted graph $(G, \mathbf{v})$ is said to be *hyperfinite* if there exists an increasing sequence of finitary percolations $(\rho_n)_{n\geq 1}$ on $(G, \mathbf{v})$ such that $\bigcup_{n\geq 1} K_{\rho_n}(\mathbf{v}) = \mathcal{V}(G)$ almost surely. It is a fact that a unimodular random planar map is hyperfinite if and only if it is a Benjamini-Schramm of finite planar maps ([AHNR18]). In particular, the UIPT is unimodular and hyperfinite.

The following corollary is a consequence of Proposition 4.2 that translates the Markov-type inequalities to a diffusivity estimate of random walks on graphs.

Corollary 4.6. *Let $(G, \mathbf{v})$ be a hyperfinite, unimodular random rooted graph with $\mathbb{E}[\deg(\mathbf{v})] < \infty$ that is almost surely planar. Let $(X_n^G)_{n\geq 0}$ be the simple random walk on G started from $\mathbf{v}$. Let ω be a vertex weighting of G so that $(G, \omega, \mathbf{v})$ is unimodular. Then*

$$\mathbb{E}\left[\deg(\mathbf{v}) \max_{1\leq m\leq n} \mathrm{dist}_\omega^G(\mathbf{v}, X_m^G)^2\right] \leq C^2 n \mathbb{E}[\deg(\mathbf{v})\omega(\mathbf{v})^2],$$

for every $n \geq 0$, where C is the universal constant from Proposition 4.2.

To sketch a proof of this corollary, we first note that by the monotone convergence theorem and a truncation, it suffices to consider any weighting ω that is almost surely bounded. Moreover, using the hyperfiniteness of $(G, \mathbf{v})$ (thus (G, ω, v)) and the dominated convergence theorem, it suffices to restrict to any finitary percolation on $(G, \omega, \mathbf{v})$, which we denote by $(G_N, \omega_N, \mathbf{v})$. A direct application of the mass transport principle shows that conditional on the isomorphism class of (G_n, ω_N), the root $\mathbf{v}$ is uniformly distributed on the vertex set of G_N. It follows that if we bias the law of $(G_N, \omega_N, \mathbf{v})$ by $\deg(\mathbf{v})$ restricted to G_N, then, conditional on the isomorphism class of (G_N, ω_N), $\mathbf{v}$ is distributed according to the stationary measure of the random walk on G_N. This allows us to apply Proposition 4.2 to obtain that

$$\mathbb{E}\left[\deg^{G_N}(\mathbf{v}) \max_{1\leq m\leq n} \mathrm{dist}_{\omega_N}^{G_N}(\mathbf{v}, X_m^{G_N})^2\right] \leq C^2 n \mathbb{E}[\deg^{G_N}(\mathbf{v})\omega_N(\mathbf{v})^2].$$

4.2 Vertex Weightings and Estimates

Recall that the coupling between the UIPT $\mathbb{M}$ and the mated CRT $\mathcal{G}^\epsilon$ in Theorem 2.1 is a rough isometry up to a polylogarithmic factor. The SLE/LQG theory then allows us to embed $\mathcal{G}^\epsilon$ into $\mathbb{C}$, where the Euclidean distance becomes much more tractable than the original graph distance on $\mathbb{M}$. Recall that given an interval $I \subset \mathbb{R}$ and $\epsilon \in (0,1)$, we have an obvious submap $\mathcal{G}^\epsilon_I$ of $\mathcal{G}^\epsilon$ induced by the vertex set $I \cap (\epsilon\mathbb{Z})$. As described in Section 2.2, the case for $\mathbb{M}$ is slightly more complicated (see Section 2.5 of [GH18] for how the "submap" $\mathbb{M}_I$ is exactly defined), but we emphasize here again the important properties that we can canonically identify $\mathbb{M}_I \setminus \partial\mathbb{M}_I$ with a subgraph of $\mathbb{M}$ and that $\mathbb{M}_I$ has a canonical root edge. For this "almost inclusion" $\iota_I : \mathbb{M}_I \to \mathbb{M}$, one can also define the corresponding functions $\phi_I : (\mathbb{M}_I) \to I \cap \mathbb{Z}$ and $\psi_I : I \cap \mathbb{Z} \to \mathcal{V}(\mathbb{M}_I)$.

Let $\mathcal{Z} = (\mathcal{L}, \mathcal{R}) : \mathbb{Z} \to \mathbb{Z}^2$ denote the two-sided two-dimensional random walk which encodes the UIPT decorated by critical site percolation via the discrete mating-of-trees bijection. Let $Z = (L, R)$ be the correlated Brownian motion defined in 1. Also recall from Section 2.1 that $\{\mathcal{G}^\epsilon\}_{\epsilon>0}$ is the set of γ^*-mated CRT maps with spacing ϵ constructed from Z, where $\gamma^* = \sqrt{8/3}$ is the special parameter corresponding to the LQG universality class of the UIPT. Let $((\mathbb{C}, h, 0, \infty), \eta)$ be the γ-quantum cone and space-filling SLE_6 curve determined by Z via the theory of mating of trees. We also assume that h is a circle average embedding (see Section 2.3 of [GH18]) and that η is parametrized by γ^*-LQG mass with respect to h, so that $\mathcal{G}^\epsilon$ is isomorphic to the adjacency graph of cells $\eta([x-\epsilon, x])$ for $x \in \epsilon\mathbb{Z}$.

Definition 4.7. For $\epsilon > 0$ and a domain $D \subset \mathbb{C}$, we denote by $\mathcal{G}^\epsilon(D)$ the subgraph of $\mathcal{G}^\epsilon$ induced by the set of vertices $x \in \epsilon\mathbb{Z}$ with $\eta([x-\epsilon, x]) \cap D \neq \emptyset$.

A crucial property of the embedding $x \mapsto \eta(x)$ is that with high probability as $\epsilon \to 0$, the maximal size of cells that intersect a fixed Euclidean ball is of order $\epsilon^{2/(2+\gamma^*)^2 + o_\epsilon(1)}$. The following lemma quantitatively describes this property, which will be useful later when we try to prove an upper bound for the Euclidean displacement of the embedded walks.

Lemma 4.8 ([GMS19])**.** *Using the above notations, for each $q \in (0, \frac{2}{(2+\gamma^*)^2})$, each $r \in (0,1)$, and each $\epsilon \in (0,1)$,*

$$\mathbb{P}[\operatorname{diam}(\eta([x-\epsilon, x]) \leq \epsilon^q, \forall x \in \mathcal{G}^\epsilon(B_r(0))] \geq 1 - \epsilon^{\alpha(q,\gamma^*) + o_\epsilon(1)},$$

where the rate of the $o_\epsilon(1)$ depends only on q, r, and γ^* and $\alpha(q, \gamma^*) := \frac{q}{2\gamma^{*2}}\left(\frac{1}{q} - 2 - \frac{\gamma^{*2}}{2}\right)^2 - 2q > 0.$

The only property of $\alpha(q, \gamma^*)$ we will use is that it tends to ∞ as $q \to 0$.

We now couple $\mathcal{G}^\epsilon$ and $\mathbb{M}$ together via Theorem 2.1 but with a specific choice of a sequence of intervals $\{I^\epsilon\}$ instead of using $\{[-n, n]_\mathbb{Z}\}$. The reason for doing so is that it allows us to more easily produce vertex weightings on $\mathcal{V}(\mathbb{M}_{I^\epsilon})$ and $\mathcal{V}(\mathcal{G}^\epsilon)$ which are reversible/unimodular, which in turn allows us to apply the Markov-type inequalities introduced in Section 4.1. We also note that the coupling we are about to define requires a slight modification of Theorem 2.1, but it easily follows by using translation invariance to transfer the interval $[-n, n]_\mathbb{Z}$ to I^ϵ. To define $\{I^\epsilon\}$, we fix a large constant $K > 1$, which will eventually be chosen depending only on γ^*. For $\epsilon \in (0,1)$, let θ^ϵ be sample uniformly from $[0, \epsilon^{-K}]$ and independently of everything else. Then we couple Z and $\mathbb{M}$ using Theorem 2.1 for the interval

$$I^\epsilon = [a^\epsilon, b^\epsilon] := [-\theta^\epsilon, \epsilon^{-K} - \theta^\epsilon] \tag{10}$$

and with $n = \lfloor \epsilon^{-K} \rfloor$. The following lemma states that the random shifting θ^ϵ makes the root edge $\mathbf{e}$ of $\mathbb{M}_{I^\epsilon}$ uniform, which will subsequently help us check that a certain vertex weighting on $\mathcal{V}(\mathbb{M}_{I^\epsilon})$ is reversible. See Section 3.1 of [GH18] for a proof.

Lemma 4.9. *The planar map $\mathbb{M}_{I^\epsilon}$ is almost surely determined by the translated random walk $(\mathcal{Z}_{t-\theta^\epsilon} - -\theta^\epsilon)_{t\in\mathbb{Z}}$. Moreover, if we condition on the random walk/Brownian motion pair*

$$((\mathcal{Z}_{t-\theta^\epsilon} - \mathcal{Z}_{-\theta^\epsilon})_{t\in\mathbb{Z}}, (Z_{t-\epsilon\theta^\epsilon} - Z_{-\epsilon\theta^\epsilon})_{t\in\mathbb{R}})$$

and on the event $\{\mathbf{e} \in \mathcal{E}(\mathbb{M}_{I^\epsilon}) \setminus \mathcal{E}(\partial\mathbb{M}_{I^\epsilon})\}$, then the conditional law of the root edge $\mathbf{e}$ is uniform on $\mathcal{E}(\mathbb{M}_{I^\epsilon}) \setminus \mathcal{E}(\partial\mathbb{M}_{I^\epsilon})$

Recall the functions $\phi_{I^\epsilon} : (\mathbb{M}_{I^\epsilon}) \to I^\epsilon \cap \mathbb{Z}$ and $\psi_{I^\epsilon} : I^\epsilon \cap \mathbb{Z} \to \mathcal{V}(\mathbb{M}_{I^\epsilon})$ corresponding to the "almost inclusion" $\iota_{I^\epsilon} : \mathbb{M}_{I^\epsilon} \to \mathbb{M}$. Define

$$\phi^\epsilon := \epsilon\phi_{I^\epsilon} : \mathcal{V}(\mathbb{M}_{I^\epsilon}) \to \mathcal{V}(\mathcal{G}^\epsilon_{I^\epsilon}), \quad \psi^\epsilon := \psi_{I^\epsilon}(\cdot/\epsilon) : \mathcal{V}(\mathcal{G}^\epsilon_{I^\epsilon}) \to \mathcal{V}(\mathbb{M}_{I^\epsilon}), \quad \Phi^\epsilon := \eta \circ \phi^\epsilon : \mathcal{V}(M_{I^\epsilon}) \to \mathbb{C}. \tag{11}$$

The map Φ^ϵ is the embedding of the map $\mathbb{M}_{I^\epsilon}$ into $\mathbb{C}$ that we will be mainly working with.

Our goal is to use the embeddings $\eta : \mathcal{V}(\mathcal{G}^\epsilon) \to \mathbb{C}$ and $\Phi : \mathcal{V}(\mathbb{M}^\epsilon_{I^\epsilon}) \to \mathbb{C}$ to produce a unimodular vertex weighting $\omega^\epsilon_{\mathcal{G}}$ on $\mathcal{G}^\epsilon$ and a reversible vertex weighting $\omega^\epsilon_{\mathbb{M}}$ on $\mathbb{M}^\epsilon_{I^\epsilon}$ to which the Markov-type inequalities can be applied. For the $\omega^\epsilon_{\mathcal{G}}$- and $\omega^\epsilon_{\mathbb{M}}$-weighted distances to be close to the corresponding Euclidean distances, it is natural to define these weightings so that

$$\omega^\epsilon_{\mathcal{G}}(x) \approx \operatorname{diam}\left(\eta([x-\epsilon, x])\right), \quad \omega^\epsilon_{\mathbb{M}}(v) \approx \operatorname{diam}\left(\eta([\phi^\epsilon(v)-\epsilon, \phi^\epsilon(v)])\right) \tag{12}$$

However, there are two issues that we encounter. First, since the law of the pair (h, η) is not exactly invariant under time translations $(h, \eta) \mapsto (h(\cdot + \eta(t)), \eta(\cdot + t) - \eta(t))$ for $t \in \mathbb{R}$, so the weightings in (12) are not reversible for $(\mathbb{M}, \mathbf{v})$ or unimodular for $(\mathcal{G}^\epsilon, 0)$. Second, the boundaries of the cells $\eta([\phi^\epsilon(v) - \epsilon, \phi^\epsilon(v)])$ and $\eta([\phi^\epsilon(v') - \epsilon, \phi^\epsilon(v')])$ corresponding to adjacent vertices v, v' in $\mathcal{V}(\mathbb{M}_{I^\epsilon})$ need not intersect, and intersecting cells need not correspond to adjacent vertices. Therefore, the Euclidean distance between the embeddings of two vertices of $\mathcal{V}(\mathbb{M}_{I^\epsilon})$ might not be comparable to the minimal sum of the diameters of the cells along a path in $\mathbb{M}_{I^\epsilon}$ connecting the two given vertices.

To circumvent the first issue, we will use a rescaled version of the translated pair $(h, \eta) \mapsto (h(\cdot + \eta(t)), \eta(\cdot + t) - \eta(t))$ whose law will be stationary in t by definition. For the second issue, note that the coupling in Theorem 2.1 guarantees that cells corresponding to adjacent vertices of $\mathbb{M}_{I^\epsilon}$ cannot lie more than polylogarithmic graph distance from each other in $\mathcal{G}^\epsilon$, so we can use the Euclidean diameter of the union of cells in a $\mathcal{G}^\epsilon$-graph distance neighborhood of polylogarithmic size to dominate the Euclidean distance between the embeddings of two adjacent vertices of $\mathcal{V}(\mathbb{M}_{I^\epsilon})$. After giving the precise definitions of these modified weights, we will establish that their corresponding weighted distances are comparable to Euclidean distances up to subpolynomial errors.

From [DMS14], we know that for each $t \in \mathbb{R}$ the shifted pair $(h, \eta) \mapsto (h(\cdot + \eta(t)), \eta(\cdot + t) - \eta(t))$ agrees in law with (h, η) modulo rotation and scaling: for each $t \in \mathbb{R}$, there exists a random constant $\rho_t \in \mathbb{C}$ such that if

$$h^t := h(\rho_t \cdot + \eta(t)) + Q \log |\rho_t|, \quad \eta^t := \rho_t^{-1}(\eta(\cdot + t) - \eta(t)) \tag{13}$$

then we have $(h^t, \eta^t) \overset{d}{=} (h, \eta)$. Choose any p greater than the exponent of the polylogarithmic factor in Theorem 2.1. For $x \in \epsilon\mathbb{Z} = \mathcal{V}(\mathcal{G}^\epsilon)$, we define the following translation-invariant weighting

$$\omega^\epsilon_{\mathcal{G}} := \max\left\{1, |\rho_x|^{-1} \operatorname{diam}\left(\bigcup_{y \in \mathcal{V}\left(\mathcal{B}^{\mathcal{G}^\epsilon}_{(\log \epsilon^{-1})^p}(x)\right)} \eta([y-\epsilon, y]) \right)\right\}. \tag{14}$$

We also define a weighting on $\mathcal{V}(\mathbb{M}_{I^\epsilon})$ by

$$\omega^\epsilon_{\mathbb{M}}(v) := \omega^\epsilon_{\mathcal{G}}(\phi^\epsilon(v)). \tag{15}$$

With these weightings, we can deduce the following result on unimodularity/reversibility.

Lemma 4.10. *The vertex-weighted graph $(\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, 0)$ is unimodular in the sense of Definition 4.3. Denote $\mathring{\mathbb{M}}_{I^\epsilon} := \mathbb{M}_{I^\epsilon} \setminus \mathcal{E}(\partial\mathbb{M}_{I^\epsilon})$. Then, conditioning on the event that the root edge $\mathbf{e}$ is in $\mathcal{E}(\mathbb{M}_{I^\epsilon}) \setminus \mathcal{E}(\partial\mathbb{M}_{I^\epsilon})$, the vertex-weighted graph $(\mathring{\mathbb{M}}_{I^\epsilon}, \omega^\epsilon_{\mathbb{M}}, \mathbf{v})$ is reversible in the sense of Definition 4.4.*

Proof. For the unimodularity of $(\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, 0)$, recall that for $x \in \mathcal{V}(\mathcal{G}^\epsilon) = \epsilon\mathbb{Z}$ we have $(h^x, \eta^x) \overset{d}{=} (h, \eta)$. Since the vertex-weighted graph $(\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, x)$ is constructed from (h^x, η^x) in the same deterministic way that $(\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, 0)$ is constructed from (h, η), we must have $(\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, x, 0) \overset{d}{=} (\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, 0, -x)$. Moreover, using the invariance of the law of Z under time reversal, we also have $(\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, 0, -x) \overset{d}{=} (\mathcal{G}^\epsilon, \omega^\epsilon_{\mathcal{G}}, 0, x)$. Then taking expectation of any nonnegative measurable function verifies the mass-transport principle.

For the reversibility of $(\mathring{\mathbb{M}}_{I^\epsilon}, \omega^\epsilon_{\mathbb{M}}, \mathbf{v})$, note that Lemma 4.9 implies that conditioning on $(\mathbb{M}_{I^\epsilon}, \omega^\epsilon_{\mathbb{M}})$ and on the event $\{\mathbf{e} \in \mathcal{E}(\mathbb{M}_{I^\epsilon}) \setminus \mathcal{E}(\partial\mathbb{M}_{I^\epsilon})\}$, the root edge $\mathbf{e}$ is uniformly distributed on $\mathcal{E}(\mathring{\mathbb{M}}_{I^\epsilon})$. It follows that under this conditioning, $\mathbf{v}$ is sampled from the uniform measure on vertices of $\mathring{\mathbb{M}}_{I^\epsilon}$ weighted by their $\mathring{\mathbb{M}}_{I^\epsilon}$-degree, so $(\mathring{\mathbb{M}}_{I^\epsilon}, \omega^\epsilon_{\mathbb{M}}, \mathbf{v})$ is reversible. (Here we use Remark 4.5 and the fact that conditionally uniform root implies unimodularity). □

In the final preparation for bounding the Euclidean displacement of the embedded walks, we provide two estimates for the weightings defined in (14) and (15). The first is a second moment estimate for the weight functions at the root vertex, which will be helpful when we apply the Markov-type inequality in Corollary 4.6 to bound the displacement with respect to the vertex-weighted graph distance.

Proposition 4.11. *Let $\omega^\epsilon_{\mathcal{G}}$ and $\omega^\epsilon_{\mathbb{M}}$ be as in (14) and (15), respectively. Then for each $\epsilon \in (0,1)$,*

$$\mathbb{E}[\omega^\epsilon_{\mathcal{G}}(0)^2 \deg^{\mathcal{G}^\epsilon}(0)] \le \epsilon^{1+o_\epsilon(1)}, \quad \mathbb{E}[\omega^\epsilon_{\mathbb{M}}(0)^2] \le \epsilon^{1+o_\epsilon(1)}.$$

The reason for the exponent of ϵ in these bounds is that the root cell $\eta([-\epsilon, 0])$ should look approximately uniform among those in $\mathcal{G}^\epsilon$ which intersect the unit disk $\mathbb{D}$. There should typically be of order ϵ^{-1} such cells, so the expected Lebesgue measure of the root cell should be of order ϵ. Then standard SLE/LQG estimates can be applied to show that, e.g., replacing the Lebesgue measure by the squared Euclidean diameter of the root cell, taking the union of cells in a ball of polylogarithmic size, and weighting by $\deg^{\mathcal{G}^\epsilon}(0)$, should not affect the exponent. The details can be found in Section 4.1 of [GH18].

The following estimate concerns the distortion factor ρ_x^{-1}, which appears in the definitions of the weightings.

Proposition 4.12. *Let ρ_t be the scaling factor defined in (13) for $t \in \mathbb{R}$. Then there exists $\alpha > 0$ such that for each $S > 1$,*

$$\mathbb{P}\left[\sup_{t \in \eta^{-1}(\mathbb{D})} |\rho_t| \le S\right] \ge 1 - O_S(S^{-\alpha})$$

and for each $\epsilon \in (0,1)$,

$$\mathbb{P}\left[\operatorname{diam}\left(\bigcup_{y \in \mathcal{V}(\mathcal{B}^{\mathcal{G}^\epsilon}_{(\log \epsilon^{-1})^p}(x))} [y-\varepsilon, y]\right) \le S\omega^\epsilon_{\mathcal{G}}(x), \forall x \in \mathcal{V}(\mathcal{B}^{\mathcal{G}^\epsilon}_{1/2}(0))\right] \ge 1 - O_S(S^{-\alpha}) - o^\infty_\epsilon(\epsilon),$$

where $o^\infty_\epsilon(\epsilon)$ means $o_\epsilon(\epsilon^s)$ for all $s \in \mathbb{R}$.

This proposition follows from estimates for the circle average process of a GFF. The details can be found in Section 4.2 of [GH18].

4.3 Upper bound on Euclidean displacement

Recall $\mathring{\mathbb{M}}_{I^\epsilon} = \mathbb{M}_{I^\epsilon} \setminus \mathcal{E}(\partial \mathbb{M}_{I^\epsilon})$ defined in Lemma 4.10. Recall also that for $\epsilon \in (0,1)$, $X^{\mathcal{G}^\epsilon}$ is a simple random walk on $\mathcal{G}^\epsilon$ on $\mathcal{G}^\epsilon$ started from 0, and $X^{\mathring{\mathbb{M}}_{I^\epsilon}}$ is a simple random walk on $\mathring{\mathbb{M}}_{I^\epsilon}$ started from the root vertex $\mathbf{v}$. Before stating the main result of this section, we first apply the Markov-type inequalities in Corollary 4.6 to give an upper bound for the displacement of the SRW with respect to the vertex-weighted graph distance. Let $\omega_{\mathcal{G}}^\epsilon$ and $\omega_{\mathbb{M}}^\epsilon$ be the weightings defined in (14) and (15). Recall also the definition of weighted graph distance in (9)

Lemma 4.13. *Denote $d_{\omega_{\mathcal{G}}^\epsilon} := \mathrm{dist}_{\omega_{\mathcal{G}}^\epsilon}^{\mathcal{G}^\epsilon}(\cdot,\cdot)$ and $d_{\omega_{\mathbb{M}}^\epsilon} := \mathrm{dist}_{\omega_{\mathbb{M}}^\epsilon}^{\mathring{\mathbb{M}}_{I^\epsilon}}(\cdot,\cdot)$. Then*

$$\mathbb{E}\left[\max_{j\in[0,n]_{\mathbb{Z}}} d_{\omega_{\mathcal{G}}^\epsilon}\left(0, X_j^{\mathcal{G}^\epsilon}\right)^2\right] \le n\epsilon^{1+o_\epsilon(1)}.$$

Moreover, write $F^\epsilon := \{\mathbf{e} \in \mathcal{E}(M_{I^\epsilon}) \setminus \mathcal{E}(\partial M_{I^\epsilon})\}$. Then the exponent K in (10) can be chosen large enough and only depending on γ^* so that $\mathbb{P}[F^\epsilon] \ge 1 - O_\epsilon(\epsilon^{100})$ and

$$\mathbb{E}\left[\mathbf{1}_{F^\epsilon} \max_{j\in[0,n]_{\mathbb{Z}}} d_{\omega_M^\epsilon}\left(\mathbf{v}, X_j^{\mathring{\mathbb{M}}_{I^\epsilon}}\right)^2\right] \le n\epsilon^{1+o_\epsilon(1)}.$$

Proof. Using Remark 4.5 and Lemma 4.10, we know that $(\mathring{\mathbb{M}}_{I^\epsilon}, \omega_{\mathbb{M}}^\epsilon, \mathbf{v})$ biased by $\deg^{\mathring{\mathbb{M}}_{I^\epsilon}}(\mathbf{v})^{-1}$ is unimodular. And since $(\mathcal{G}^\epsilon, \omega_{\mathcal{G}}^\epsilon, 0)$ is also unimodular by Lemma 4.10, we can apply Corollary 4.6 to see that there exists a universal constant $C > 0$ such that for every $\epsilon \in (0,1)$ and every $n \in \mathbb{N}$,

$$\mathbb{E}\left[\max_{j\in[0,n]_{\mathbb{Z}}} d_{\omega_{\mathcal{G}}^\epsilon}\left(0, X_j^{\mathcal{G}^\epsilon}\right)^2 \deg^{\mathcal{G}^\epsilon}(0)\right] \le nC^2\mathbb{E}[\omega_{\mathcal{G}}^\epsilon(0)^2 \deg^{\mathcal{G}^\epsilon}(0)] \tag{16}$$

and

$$\mathbb{E}\left[\max_{j\in[0,n]_{\mathbb{Z}}} d_{\omega_M^\epsilon}\left(\mathbf{v}, X_j^{\mathring{\mathbb{M}}_{I^\epsilon}}\right)^2 \middle| F^\epsilon\right] \le nC^2\mathbb{E}[\omega_{\mathbb{M}}^\epsilon(\mathbf{v})^2 | F^\epsilon]. \tag{17}$$

Combining (16) and (17) with the second moment estimates for the weight functions at the root vertex provided in Proposition 4.11 proves the lemma, except that we have to get rid of the conditioning in (17). This can be done by choosing K large enough so that $\mathbb{P}[F^\epsilon] \ge 1 - O_\epsilon(\epsilon^{100})$. The details for estimating this probability can be found in Lemma 1.11 of [GHS20] and Lemma 3.7 of [GH18]. □

The following proposition provides an upper bound on the Euclidean displacement the walks $X^{\mathcal{G}^\epsilon}$ and $X^{\mathring{\mathbb{M}}_{I^\epsilon}}$. By taking the parameters ζ and $\widehat{\zeta}$ to be small, we can interpret the proposition to mean that the embedded walks typically take time at least $\epsilon^{-1+o_\epsilon(1)}$ to exit $\mathbb{D}$. Recall the embedding Φ^ϵ of $\mathbb{M}_{I^\epsilon}$ into $\mathbb{C}$ defined in (11).

Proposition 4.14. *For each $\zeta, \widehat{\zeta} \in (0,1)$ with $2\zeta < \widehat{\zeta}$, there exists $\alpha = \alpha(\zeta, \widehat{\zeta}, \gamma^*) > 0$ such that for each $\epsilon \in (0,1)$,*

$$\mathbb{P}\left[\max_{j\in[0,\epsilon^{-1+\widehat{\zeta}}]_{\mathbb{Z}}} \left|\eta(X_j^{\mathcal{G}^\epsilon})\right| \le \epsilon^\zeta\right] \ge 1 - O_\epsilon(\epsilon^\alpha)$$

and

$$\mathbb{P}\left[\max_{j\in[0,\epsilon^{-1+\widehat{\zeta}}]_{\mathbb{Z}}} \left|\Phi^\epsilon\left(X_j^{\mathring{\mathbb{M}}_{I^\epsilon}}\right)\right| \le \epsilon^\zeta\right] \ge 1 - O_\epsilon(\epsilon^\alpha).$$

Proof. The proof comes in two steps. First, we define a good event that happens with probability $1-O_\epsilon(\epsilon^\alpha)$. Second, we show that on this event, the Euclidean distances can be bounded above in terms of the $\omega^\epsilon_{\mathcal{G}}$- and $\omega^\epsilon_{\mathbb{M}}$-distances, so that Lemma 4.13 can be applied.

Step 1: definition of a good event. Let $q := \frac{1}{(2+\gamma^*)^2}$. Let $\zeta, \widehat{\zeta} \in (0,1)$ with $2\zeta < \widehat{\zeta}$. Set $\delta := (\zeta \wedge (\widehat{\zeta} - 2\zeta))/100$. Let $E^\epsilon = E^\epsilon(\zeta, \widehat{\zeta}, q)$ be the intersection of the following four events.

1. Recall the notation $\mathcal{G}^\epsilon(B_{1/2}(0))$ in Definition 4.7. For all $x \in \mathcal{V}(\mathcal{G}^\epsilon(B_{1/2}(0)))$ we have
$$\operatorname{diam}\left(\bigcup_{y \in \mathcal{V}\left(\mathcal{B}^{\mathcal{G}^\epsilon}_{(\log \epsilon^{-1})^p}(x)\right)} \eta([y-\epsilon, y])\right) \le \epsilon^{-\delta} \omega^\epsilon_{\mathcal{G}}(x).$$
2. Recall the notations in Lemma 4.13. We have
$$\max_{j \in [0, \epsilon^{-1+\widehat{\zeta}}]_{\mathbb{Z}}} d_{\omega^\epsilon_{\mathcal{G}}}\left(0, X^{\mathcal{G}^\epsilon}_j\right) \le \epsilon^{\zeta+2\delta}, \qquad \max_{j \in [0, \epsilon^{-1+\widehat{\zeta}}]_{\mathbb{Z}}} d_{\omega^\epsilon_{\mathbb{M}}}(\mathbf{v}, X^{\mathring{\mathbb{M}}_{I^\epsilon}}_j) \le \epsilon^{\zeta+2\delta}.$$
3. Each cell $\eta([x-\epsilon, x])$ for $x \in \mathcal{V}(\mathcal{G}^\epsilon(B_{1/2}(0)))$ has Euclidean diameter at most ϵ^q.
4. The coupling conditions in Theorem 2.1 with the intervals I^ϵ defined in (10).

By applying Proposition 4.12 with $S = \epsilon^{-\delta}$, condition 1 can be satisfied except on an event of probability decaying faster than a positive power of ϵ. The same is true for condition 2 by using Lemma 4.13 with $n = \lfloor \epsilon^{-1+\widehat{\zeta}} \rfloor$ and applying Markov inequality (noting that $\widehat{\zeta} > 2\zeta + 4\delta$). Using Lemma 4.8 and our choice of coupling, respectively, we can also guarantee conditions 3 and 4. Therefore, there exists $\alpha = \alpha(\zeta, \widehat{\zeta}, \gamma^*) > 0$ such that $\mathbb{P}[E^\epsilon] \ge 1 - O_\epsilon(\epsilon^\alpha)$.

Step 2: Comparison of Euclidean and graph distances. Assume E^ϵ occurs, and we will show that on this event, the Euclidean distances of the embedded walks can be compared to the $d_{\omega_{\mathcal{G}^\epsilon}}$- and $d_{\omega^\epsilon_{\mathbb{M}}}$-distances. The analysis for both cases is almost the same, and we will prove the harder case of $d_{\omega^\epsilon_{\mathbb{M}}}$.

Let $v \in \mathcal{V}(\mathbb{M}_{I^\epsilon})$ be any vertex so that $\Phi^\epsilon(v) \in \mathbb{D}$ and let $P^{\mathbb{M}} : [0, |P^{\mathbb{M}}|]_{\mathbb{Z}} \to \mathcal{V}(\mathbb{M})$ be any path in $\mathring{\mathbb{M}}_{I^\epsilon}$ from $\mathbf{v}$ to v. Note that the embedded path $\Phi^\epsilon(P^{\mathbb{M}})$ may not be contained in $B_{1/2}(0)$, so we first work with a portion of $P^{\mathbb{M}}$ that is entirely contained in $B_{1/2}(0)$, in which case condition 1 applies. It turns out that doing so gives an upper bound for $|\Phi^\epsilon(v)|$ only up to a constant multiple. To this end, let i_* be the smallest i such that $\Phi^\epsilon(P^{\mathbb{M}}(i_*+1)) \notin B_{1/2}(0)$, or let $i_* = |P^{\mathbb{M}}|$ if the entire path $\Phi^\epsilon(P^{\mathbb{M}})$ is contained in $B_{1/2}(0)$. Write $v_* := \Phi^\epsilon(P^{\mathbb{M}}(i_*))$. Note that by condition 3 we have either $v_* = v$ or $|\Phi^\epsilon(v_*)| \ge 1/2 - o_\epsilon(1)$. Since $|\Phi^\epsilon| \le 1$, for ϵ small enough we have $|\Phi^\epsilon(v)| \le 4|\Phi^\epsilon(v_*)|$.

Recall that the coupling $\phi_\epsilon : V(\mathbb{M}_{I^\epsilon}) \to \mathcal{V}(\mathcal{G}^\epsilon_{\epsilon I^\epsilon})$ in Theorem 2.1 guarantees a path of length most $(\log \epsilon^{-K})^p$ between each pair of vertices $\phi^\epsilon(P^{\mathbb{M}}(i-1))$ and $\phi^\epsilon(P^{\mathbb{M}}(i))$. Concatenating these paths and noting that p is chosen to be larger than the exponent of the polylogarithmic factor in Theorem 2.1, we obtain a path $P^{\mathcal{G}^\epsilon}$ in $\mathcal{G}^\epsilon$ from 0 to $\phi^\epsilon(v^*)$ satisfying
$$P^{\mathcal{G}^\epsilon}([0, |P^{\mathcal{G}^\epsilon}|_{\mathbb{Z}}]) \subset \bigcup_{i=1}^{i_*} \mathcal{B}^{\mathcal{G}^\epsilon}_{(\log \epsilon^{-1})^p}(\phi^\epsilon(P^{\mathbb{M}}(i))).$$

Since consecutive vertices in $P^{\mathcal{G}^\epsilon}$ correspond to adjacent cells, the above containment implies the upper bound
$$\Phi^\epsilon(v_*) \le \sum_{i=1}^{i_*} \operatorname{diam}\left(\bigcup_{y \in \mathcal{B}^{\mathcal{G}^\epsilon}_{(\log \epsilon^{-1})^p}(\phi^\epsilon(P^{\mathbb{M}}(i)))} \eta([y-\epsilon, y])\right).$$

Since $\eta(\phi^\epsilon(P^{\mathbb{M}}(i))) = \Phi^\epsilon(P^{\mathbb{M}}(i)) \in B_{1/2}(0)$ for $i \in [0, i_*]_{\mathbb{Z}}$, we can apply the bound in condition 1 to see that

$$|\Phi^\epsilon(v_*)| \le \epsilon^{-\delta} \sum_{i=1}^{i_*} \omega^\epsilon_{\mathcal{G}}(\phi^\epsilon(P^{\mathbb{M}}(i))) = \epsilon^{-\delta} \sum_{i=1}^{i_*} \omega^\epsilon_{\mathbb{M}}(P^{\mathbb{M}}(i)).$$

Taking infimum over all paths $P^{\mathbb{M}}$ from $\mathbf{v}$ to v in $\mathring{M}_{I^\epsilon}$ we have $|\Phi^\epsilon(v_*)| \le \epsilon^{-\delta} d_{\omega^\epsilon_{\mathbb{M}}}(\mathbf{v}, v)$. Combining with conditions 2 and 3 implies that if we apply this to any $v = X_j^{\mathring{M}_{I^\epsilon}}$ for $j \in [0, \epsilon^{-1+\widehat{\zeta}}]_{\mathbb{Z}}$, then for ϵ small enough we have $v_* = v$. It follows that

$$\max_{j \in [0, \epsilon^{-1+\widehat{\zeta}}]_{\mathbb{Z}}} |\Phi^\epsilon(X_j^{\mathring{M}_{I^\epsilon}})| \le \epsilon^{-\delta} \cdot \max_{j \in [0, \epsilon^{-1+\widehat{\zeta}}]_{\mathbb{Z}}} d_{\omega^\epsilon_{\mathbb{M}}}(\mathbf{v}, X_j^{\mathring{\mathbb{M}}_{I^\epsilon}}) \le \epsilon^{\zeta+\delta},$$

as desired. □

4.4 Conclusion of the proof

In this section, we deduce Theorem 4.1 from Theorem 2.2 and Proposition 4.14. The overarching idea is the following: to bound $\mathring{M}_{I^\epsilon}$-graph distances, it suffices to consider $\mathcal{G}^\epsilon$-distances by the coupling in Theorem 2.1. Then Theorem 2.2 allows us to compare $\mathcal{G}^\epsilon$-distances to embedded Euclidean distances, so that Theorem 4.14 can be applied.

Proof of Theorem 4.1 Using Theorem 2.2 and Proposition 4.14, for each $\delta \in (0,1)$, there exists $\alpha = \alpha(\delta, \gamma^*) > 0$ such that with probability $1 - O_\epsilon(\epsilon^\alpha)$, the following events hold

$$\mathcal{G}^\epsilon(B_{1/2}(0)) \subset \mathcal{B}^\epsilon_{\epsilon^{-1/4-\delta}}(0), \quad \max_{j \in [0, \epsilon^{-1+\delta}]_{\mathbb{Z}}} \left| \eta\left(X_j^{\mathcal{G}^\epsilon}\right)\right| \le \frac{1}{2}, \quad \max_{j \in [0, \epsilon^{-1+\delta}]_{\mathbb{Z}}} \left| \Phi^\epsilon\left(X_j^{\mathring{M}_{I^\epsilon}}\right)\right| \le \frac{1}{2}.$$

Recall the functions $\phi^\epsilon : \mathcal{V}(\mathbb{M}_{I^\epsilon}) \to \mathcal{V}(\mathcal{G}^\epsilon_{I^\epsilon})$, $\psi^\epsilon : \mathcal{V}(\mathcal{G}^\epsilon_{I^\epsilon}) \to \mathcal{V}(\mathbb{M}_{I^\epsilon})$, and $\Phi^\epsilon = \eta \circ \phi^\epsilon$ defined in (11). If the above events occur, then

$$\phi^\epsilon\left(X_j^{\mathring{M}_{I^\epsilon}}([0, \epsilon^{-1+\delta}]_{\mathbb{Z}})\right) \subset \mathcal{G}^\epsilon(B_{1/2}(0)) \subset \mathcal{B}^{\mathcal{G}^\epsilon}_{\epsilon^{-1/4-\delta}}(0).$$

Using condition 2 of Theorem 2.1, with probability $1 - o^\infty_\epsilon(\epsilon)$ we have that

$$\psi^\epsilon\left(\mathcal{B}^{\mathcal{G}^\epsilon}_{\epsilon^{-1/4-\delta}}(0)\right) \subset \mathcal{B}^{M_{I^\epsilon}}_{\epsilon^{-1/4-2\delta}-1}(\mathbf{v}) \quad \Longrightarrow \quad \psi^\epsilon\left(\mathcal{B}^{\mathcal{G}^\epsilon}_{\epsilon^{-1/4-\delta}}(0)\right) \subset \mathcal{B}^{\mathring{M}_{I^\epsilon}}_{\epsilon^{-1/4-2\delta}}(\mathbf{v}).$$

Combining the two inclusions above and applying condition 3 of Theorem 2.1 we obtain that

$$\mathbb{P}\left[\max_{j \in [0, \epsilon^{-1+\delta}]_{\mathbb{Z}}} \mathrm{dist}^{\mathring{M}_{I^\epsilon}}\left(\mathbf{v}, X_j^{\mathring{M}_{I^\epsilon}}\right) \le \epsilon^{-1/4-3\delta}\right] \ge 1 - O_\epsilon(\epsilon^\alpha). \tag{18}$$

We would like (18) to hold for M in addition to $\mathring{M}_{I^\epsilon}$. This can be done by applying Lemma 1.11 of [GHS20] to choose an exponent K in (10) large enough, depending only on γ^*, so that with probability at least $1 - O_\epsilon(\epsilon)$, the "almost inclusion" $\iota_\epsilon : M_{I^\epsilon} \to M$ restricts to a graph isomorphism from $\mathcal{B}^{\mathring{M}_{I^\epsilon}}_{\epsilon^{-1}}(\mathbf{v})$ to $\mathcal{B}^M_{\epsilon^{-1}}(\mathbf{v})$. Since $X^{\mathring{M}_{I^\epsilon}}$ cannot leave $\mathcal{B}^{\mathring{M}_{I^\epsilon}}_{\epsilon^{-1}}(\mathbf{v})$ in fewer than ϵ^{-1} steps, we see that (18) holds for M as well. Choosing $\epsilon \in (0,1)$ with $\epsilon^{-1+\delta} = n$ and $\delta \in (0,1)$ small enough, depending only on ζ and γ^*, with $\epsilon^{-1/4-3\delta} \le n^{1/4+\zeta}$ concludes the proof. □

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